ASTR 415

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November 24, 2022

Problem Set #5

1

Write a program to integrate any number of coupled differential equations using the Euler method, fourth-order Runge-Kutta, and Leapfrog (note: Leapfrog only applies to special cases). You will be using this program in a future assignment, so make sure it's well documented. It's recommended that you use double precision throughout.

1. Use your program to solve the following differential equation for $x(t)$:

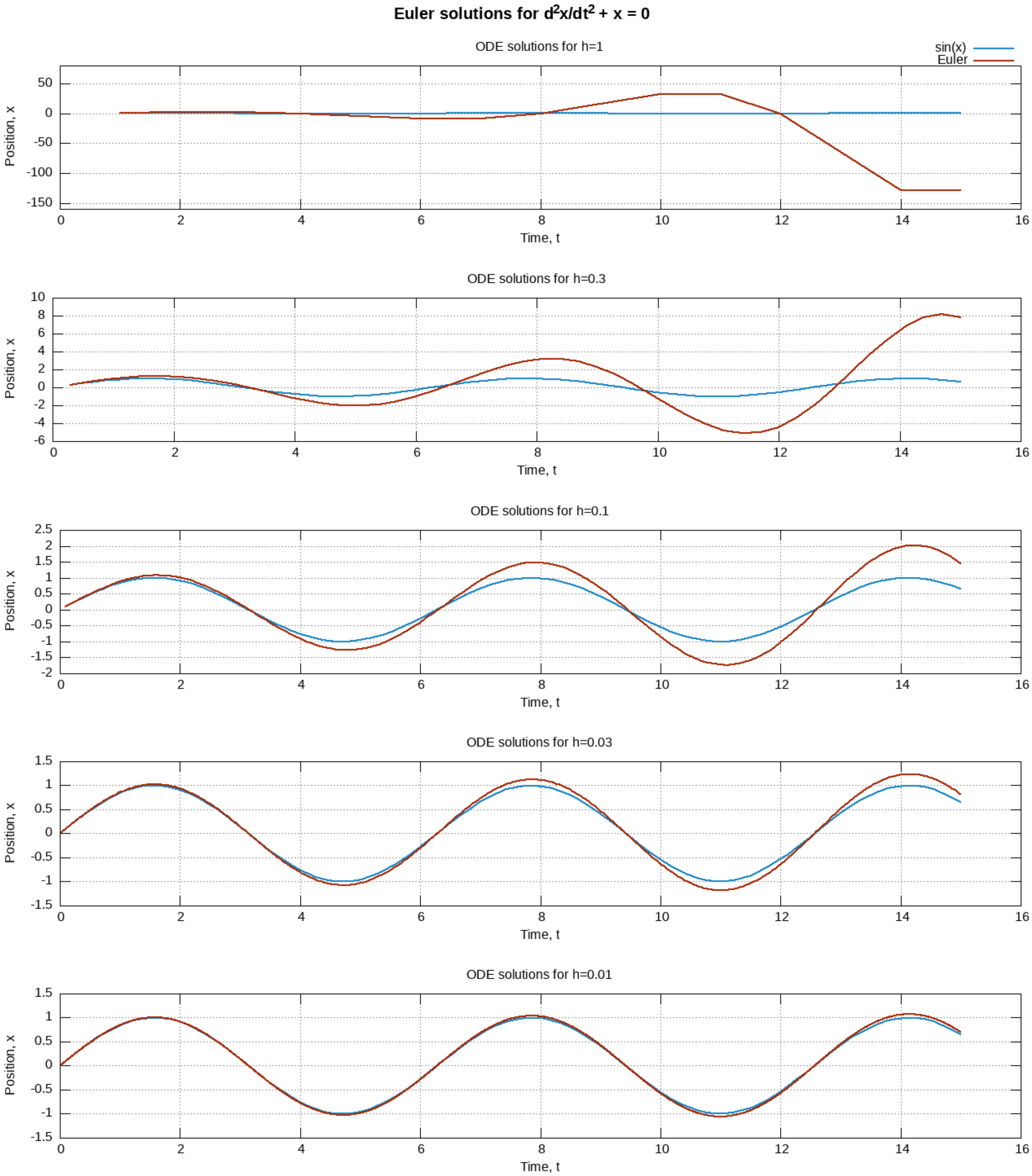
$$\frac{d^2x}{dt^2} + x = 0,$$

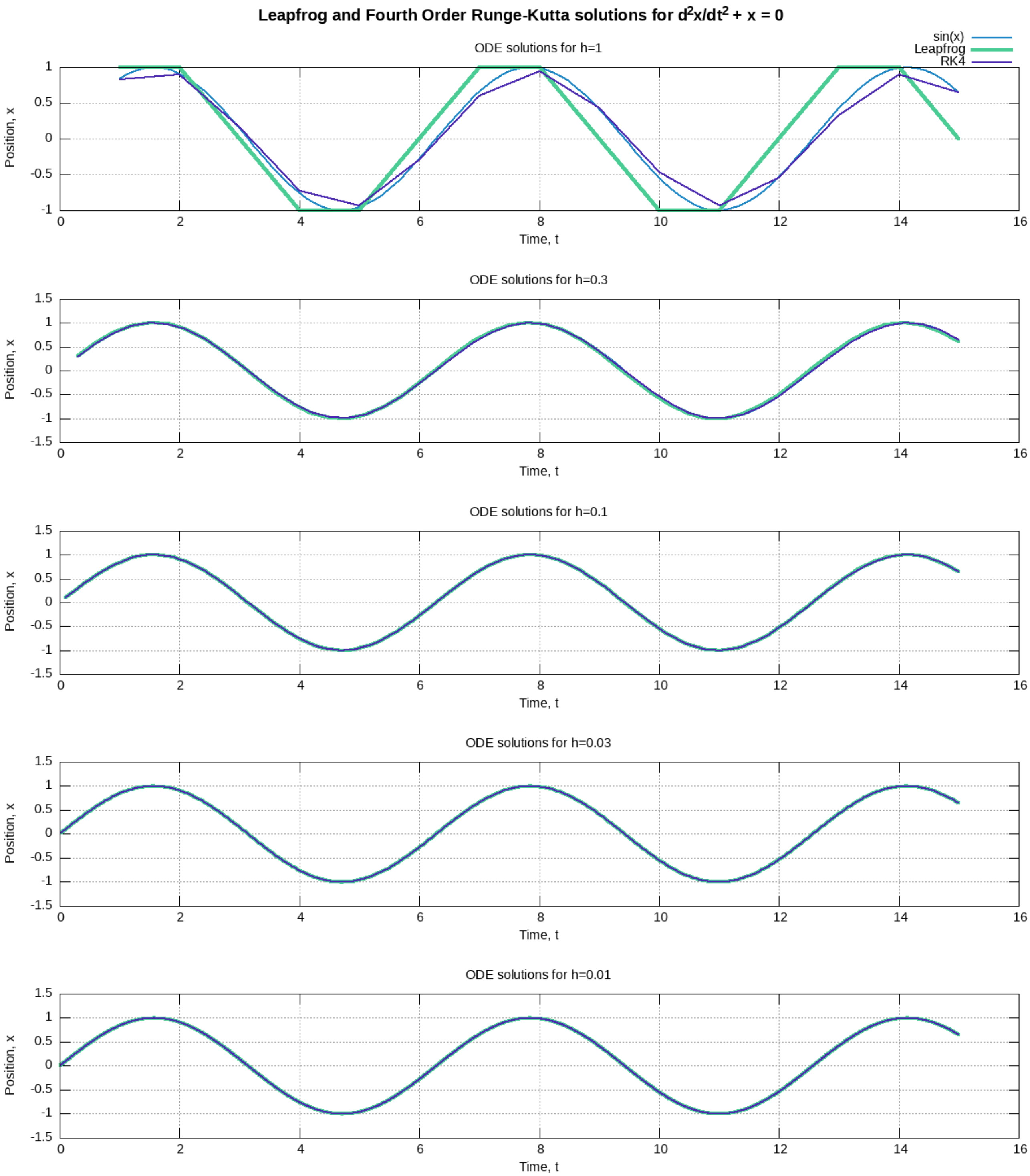
with initial conditions $x(0) = 0$, $\dot{x}(0) = 1$. Note the analytical solution is $x = \sin(t)$.

(a) Integrate the equation for $0 \leq t \leq 15$ using each of the methods, and step sizes of 1, 0.3, 0.1, 0.03, and 0.01.

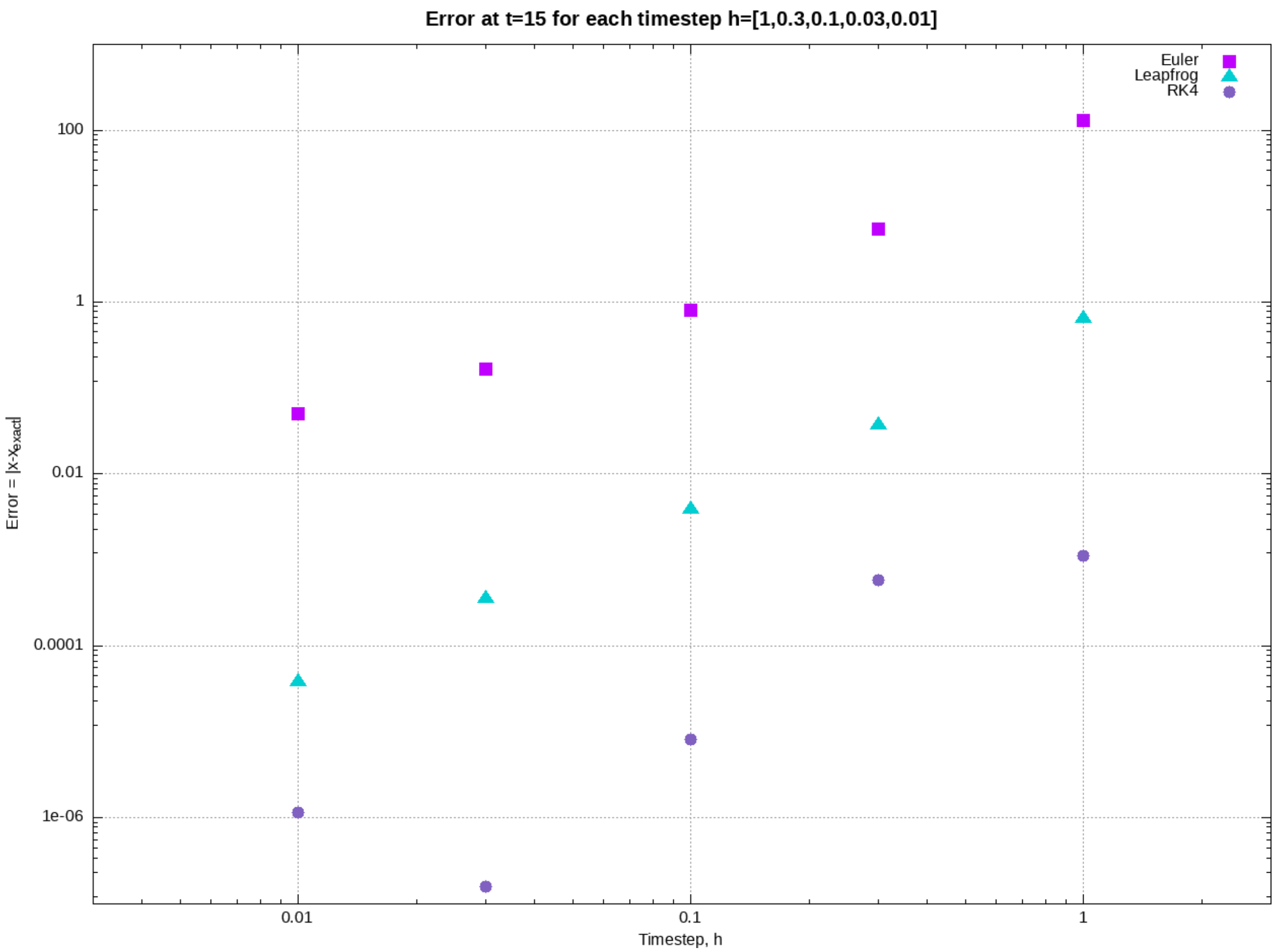
(b) Plot your integration results against the analytical solution for each case. (*Hint*: do all the Euler plots on one page, with one plot per timestep; then all the Leapfrog plots on another page, etc.) Comment on the results.

(c) Plot $\log|x_{\mathrm{numerical}}(15) - x_{\mathrm{exact}}(15)|$ as a function of $\log(\mathit{stepsize})$ in each case and comment. (*Hint*: does the error have the expected dependence on the stepsize? Remember you're integrating over many steps, not just one.)


Euler solutions for d2x/dt2 + x = 0
ODE solutions for h=1
sin(x)
Euler
Position, x
Time, t
ODE solutions for h=0.3
Position, x
Time, t
ODE solutions for h=0.1
Position, x
Time, t
ODE solutions for h=0.03
Position, x
Time, t
ODE solutions for h=0.01
Position, x
Time, t



Leapfrog and Fourth Order Runge-Kutta solutions for d2x/dt2 + x = 0
sin(x)
Leapfrog
RK4
ODE solutions for h=1
ODE solutions for h=0.3
ODE solutions for h=0.1
ODE solutions for h=0.03
ODE solutions for h=0.01
Position, x
Time, t


Error at t=15 for each timestep h=[1,0.3,0.1,0.03,0.01]
Euler
Leapfrog
RK4
100
1
0.01
0.0001
1e-06
Error = |x-x_exact|
0.01
0.1
1
Timestep, h

2

2. Now try the two-dimensional orbit described by the potential:

$$\Phi = -\frac{1}{\sqrt{1+2x^2+2y^2}},$$

where we are assuming unit mass for the particle in this potential. Show analytically that the orbits are given by the coupled differential equations:

$$\frac{d^2x}{dt^2} = -\frac{2x}{(1+2x^2+2y^2)^{3/2}}$$

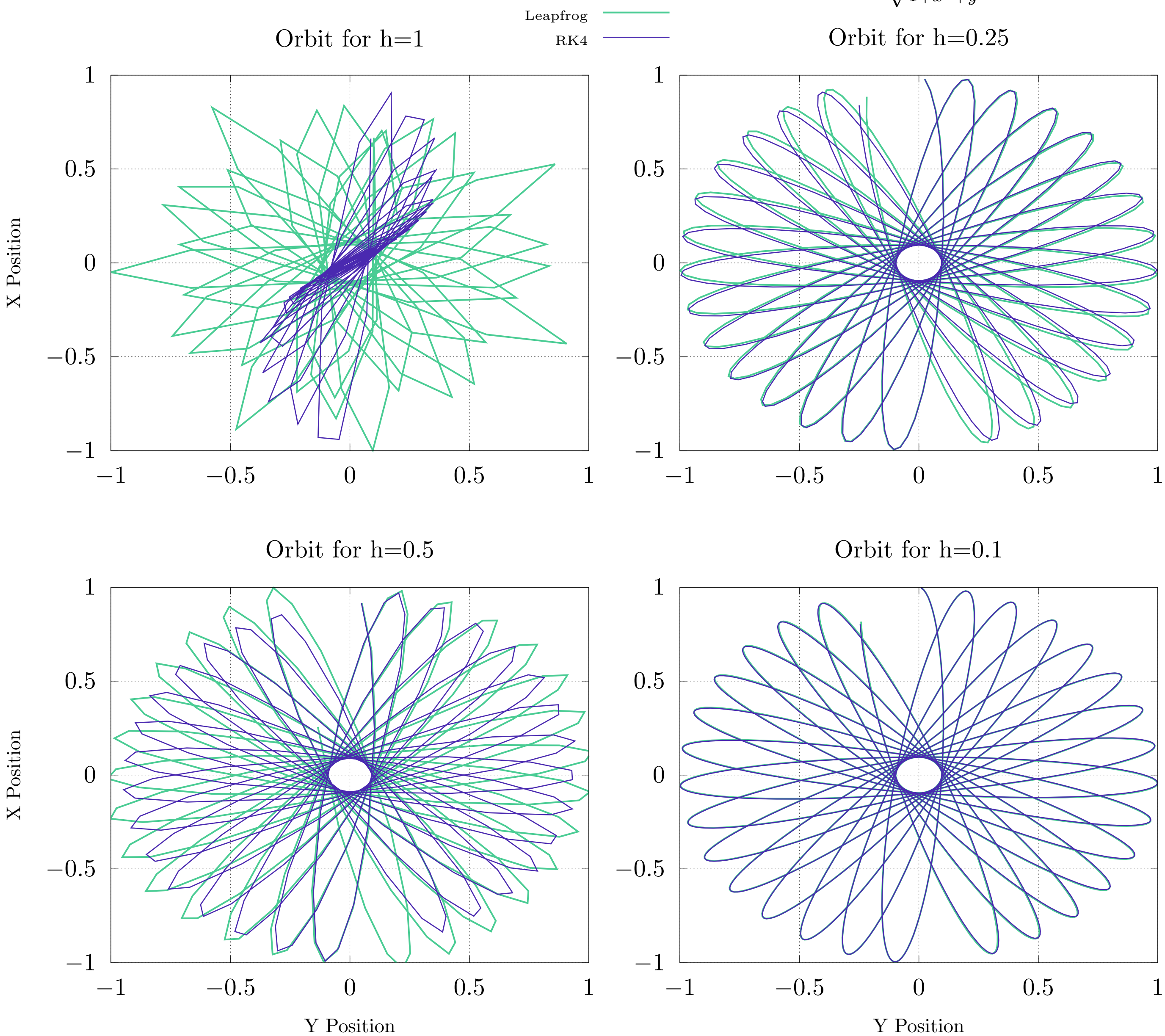
$$\frac{d^2y}{dt^2} = -\frac{2y}{(1+2x^2+2y^2)^{3/2}}$$

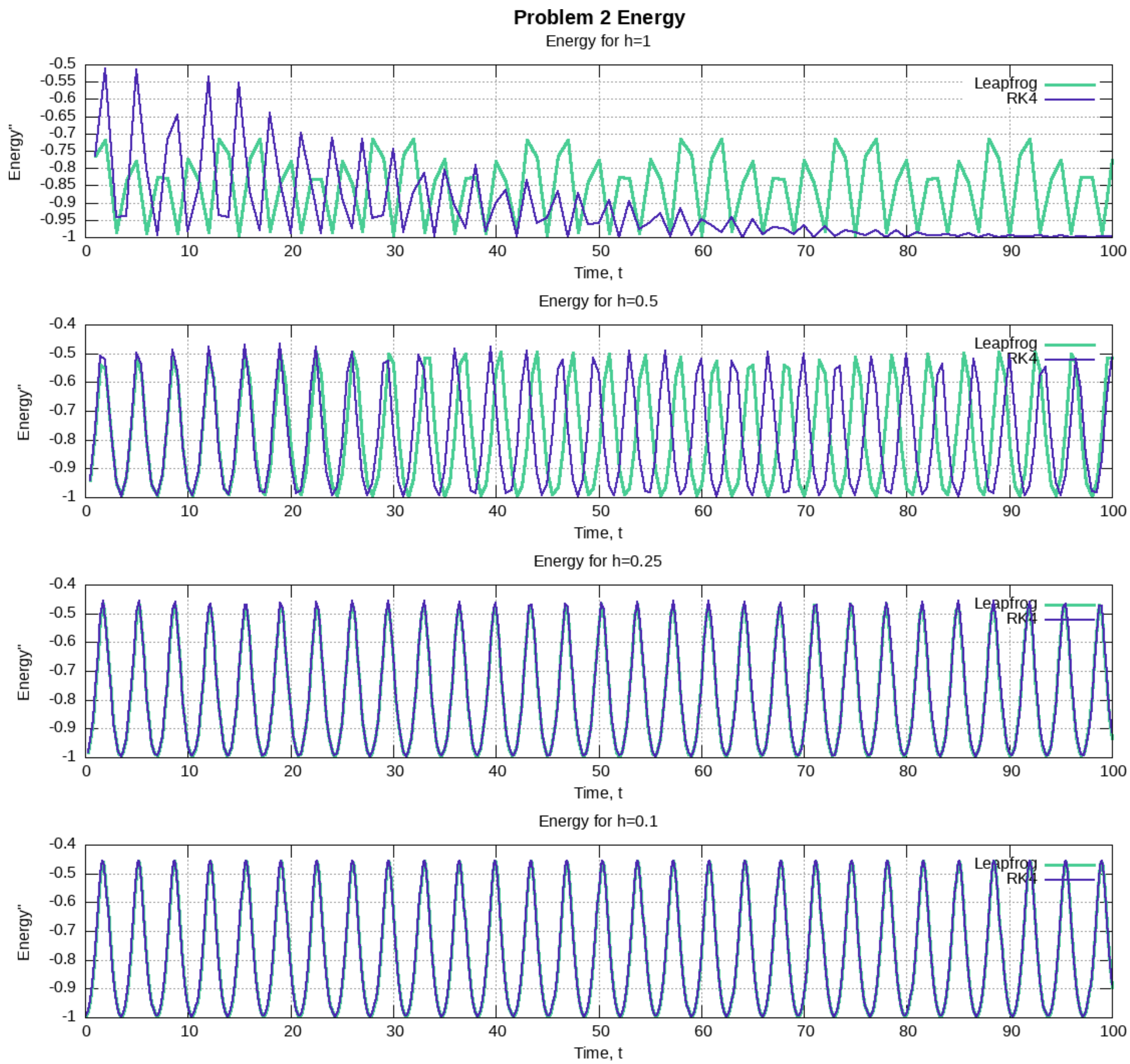
and then reduce these to 4 coupled first-order equations.

(a) Integrate this system for $0 \le t \le 100$ with the initial conditions $x = 1$, $y = 0$, $\dot{x} = 0$, $\dot{y} = 0.1$. Try this with Leapfrog and Runge-Kutta, and step sizes of 1, 0.5, 0.25, and 0.1. Plot x *vs.* y for these integrations.

(b) Plot the energy $E = (\dot{x}^2+\dot{y}^2)/2+\Phi(x,y)$ as a function of time for your integrations.

Problem 2: Two-Dimensional Orbital Trajectories for $\phi = -\frac{1}{\sqrt{1+x^2+y^2}}$




Problem 2 Energy
Energy for h=1
Energy"
Time, t
Leapfrog
RK4
Energy for h=0.5
Energy"
Time, t
Leapfrog
RK4
Energy for h=0.25
Energy"
Time, t
Leapfrog
RK4
Energy for h=0.1
Energy"
Time, t
Leapfrog
RK4